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Internal Modes of Relativistic Solitons

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We apply a linear perturbation analysis to investigate the relationship between soliton oscillations and the integrability of nonlinear PDEs in bi-dimensional spacetime. For this purpose, we consider a localized solution of the nonlinear differential equation, and study small amplitude fluctuations around it. The linearized equation is a Schrödinger-like, eigenvalue problem. By considering several nonlinear PDEs, which are known to have soliton and solitary wave solutions, we find that in systems which are integrable, this eigenvalue equation has one and only one bound state with zero frequency. Non-integrable equations—in contrast—show extra bound states. The time evolution of the oscillations are also calculated, using a numerical program to integrate the time-dependent equation. The behavior of the modes are studied, using the Fourier transform of the evolving solutions.

KEY WORDS: nonlinear PDEs; solitons; integrability.

1. INTRODUCTION

The subject of integrability of nonlinear PDEs is one of the most important topics in nonlinear physics (Das, 1989). Integrability is important, because, for such a system, we gain complete information about the evolution of the system, provided that we have the initial conditions. If one can prove that a dynamical system is integrable, one becomes sure that its solutions are free from any chaotic behavior (Berry, 1978). Moreover, in order to solve a nonlinear PDE via inverse scattering transform, one needs to know that the system is integrable (Drazin and Johnson, 1989).

A continuous integrable system has an infinite number of conserved quantities. This makes soliton solutions of such equations retain their shape and velocities after collisions with other solitons or localized perturbing potentials. Checking the integrability of nonlinear equations, and thus proving that the soliton-like solutions

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of such equations are genuine solitons, is usually a cumbersome and involved problem. Although numerical simulations help very much in monitoring the behavior of solitons during their collisions with other solitons or localized inhomogeneities (Riazi, 1993), but they cannot be considered as an exact means to check the integrability, because of the unavoidable numerical errors.

A Hamiltonian system whose phase space is $2N$ dimensional is integrable by the method of quadratures if and only if there exist exactly N functionally independent conserved quantities which are in involution, that is, the Poisson brackets of these conserved quantities with one another vanish (Das, 1989). Although this definition is for a dynamical system with a finite number of degrees of freedom, the same goes through for a system with an infinite number of degrees of freedom with a little modification. It is clear that a system with an infinite number of degrees of freedom, i.e., a continuous system, must have an infinite number of conserved quantities.

In this paper, we apply a linear perturbation analysis to investigate the relationship between the spectrum of these perturbations and integrability. It turns out that the solitons of integrable systems have only one bound state with zero frequency, which exactly corresponds to the translational symmetry of the system. Non-integrable systems, on the other hand, have extra bound states, which can be excited in the process of collision with other solitons or scattering by perturbing potentials. Furthermore, we will calculate the time evolution of the perturbations, by integrating, numerically, the time-dependent equations. The time-dependent solutions are then Fourier-transformed to find the corresponding modes and their evolution.

There are various methods for determining the integrability of a continuous system. Each of these methods has its own advantages and disadvantages. Here, we will only introduce two methods known as the Lax (Lax, 1968) and Painleve methods (Ramani *et al.*, 1989).

1.1. Lax Method

In this method, one should introduce a couple of linear operators which are called Lax pairs, L and M, by inspection. They operate on elements of a Hilbert space, and depend upon p and q. L and M should be chosen so that they satisfy Hamilton's equations and also

$$L_t = ML - LM = [M, L] \tag{1}$$

where $L_t = dL/dt$ and $[M, L]$ is the commutator of the operators M and L (Lax, 1968). By using equation (1) and induction, it is easy to show that

$$\frac{dL^n}{dt} = [M, L^n], \tag{2}$$

where n is a positive integer. Now, if we take a trace from both sides of the above equation, we obtain

$$\mathrm{tr}\frac{dL^n}{dt} = \frac{d}{dt}\mathrm{tr}L^n = \mathrm{tr}(ML^n - L^n M) = 0. \tag{3}$$

Therefore, $\mathrm{tr}L^n$ for every positive integer n are constants of motion. So, if we can introduce a Lax pair for a dynamical equation, then we can find its constants of motion, and therefore, the system will be integrable.

1.2. Painleve Method

Nonlinear differential equations may have solutions which have movable singularities, i.e., their position only depend on the arbitrary constants of integration. It is convenient to differentiate between poles and all other singularities. A *critical point* is a singularity, i.e., a point at which the solution is not analytic. This point is not a pole. Thus a critical point might be a branch point or an essential singularity. We refer to the absence of movable critical points for an ordinary differential equation as the *Painleve property*.

Painleve and Gambier (Ince, 1927) considered 50 different cases for the second order differential equation

$$\frac{d^2w}{dz^2} = F\left(z, w, \frac{dw}{dz}\right), \tag{4}$$

where F is rational in w and $\frac{dw}{dz}$, and analytic in z. Solutions of six equations of this category are not well known function. They are called *Painleve transcendents*, and are labeled as P-I to P-VI. These equations are known to be irreducible.

The 50 equations mentioned above are the only rational second order equations which satisfy the Painleve property.

According to the Painleve conjecture, *a nonlinear partial differential equation is completely integrable, if every ordinary differential equation derived from it (by exact reduction) satisfies the Painleve property.*

Such an ordinary differential equation may result from a nonlinear partial differential equation by transformations.

2. INTEGRABILITY AND SOLITON OSCILLATIONS

In this section, we apply a linear perturbation analysis to investigate the relationship between the integrability of a nonlinear PDE and the spectrum of the bound states of the linearized equation. For this purpose, we first consider a static, localized solution of the nonlinear differential equation. Let us represent it as $\phi = \phi_0(x)$. We now study small fluctuations around this static configuration

$$\phi = \phi_o + \psi(x)e^{i\omega t} \quad \text{where} \quad |\psi| \ll |\phi_o|. \tag{5}$$

Inserting this into the nonlinear differential equation and expanding to the first order in ψ, we obtain an eigenvalue equation for ψ. Subsequently, we will use the full, time-dependent equations and will integrate them numerically, to observe the time evolution of small and large amplitude oscillations. We will Fourier transform the solutions in the time domain to see the excitation and decay of the characteristic modes.

In the following sections, we apply this procedure to several well-known equations which have relativistic covariance.

2.1. Sine-Gordon Equation

As a first example, consider the well-known sine-Gordon (SG) equation:

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} = \sin \phi. \tag{6}$$

This equation is known to have the following static single soliton solution:

$$\phi_o(x) = 4 \tan^{-1} e^x. \tag{7}$$

Let us perturb this static solution:

$$\phi(x, t) = \phi_o(x) + \psi(x)e^{i\omega t}. \tag{8}$$

Inserting (8) into the sine-Gordon equation and expanding to the first order in ψ, we obtain the Schrödinger-like equation

$$-\frac{d^2}{dx^2}\psi + V(x)\psi = E\psi. \tag{9}$$

where

$$V(x) = -2\,\mathrm{sech}^2(x), \tag{10}$$

and

$$E = -1 + \omega^2. \tag{11}$$

Fortunately, this equation can be solved exactly (Lamb, 1980). It has only one bound state, which corresponds to $E = -1$ or $\omega = 0$. The corresponding eigenfunction is

$$\psi_1 = a\,\mathrm{sech}(x). \tag{12}$$

Figure 1 shows the Fourier transform of the time-dependent solution $\phi(x, t)$, corresponding to the perturbed, single-soliton solution of the SG equation. It is seen that no non-zero modes are present.

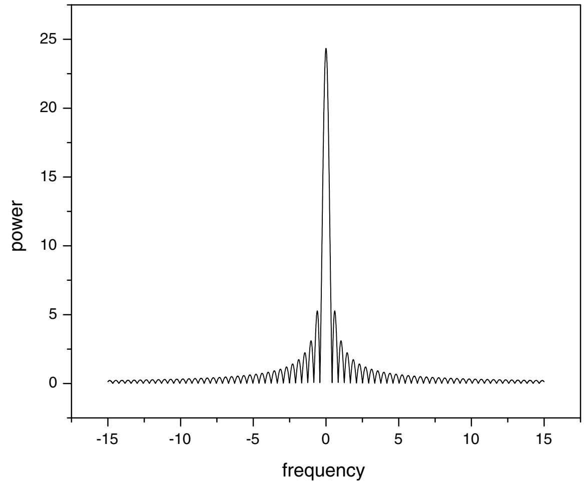

Fig. 1. Fourier transform of the time-dependent SG equation, initially perturbed by $\psi(x)$ with $a = 1$.

It can be easily shown that this eigenfunction is associated with an infinitesimal translation of the static soliton:

$$\psi_1 \simeq \phi_o(x + \delta x) - \phi_o(x). \tag{13}$$

Another solution to equation (9) is

$$\psi_2 = x\,\mathrm{sech}(x) + \sinh(x), \tag{14}$$

but this solution does not satisfy the necessary boundary conditions. Bound states are required to be localized and bounded.

2.2. The ϕ^4 System

The well-known ϕ^4 system with the dynamical equation

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = \phi - \phi^3 \tag{15}$$

has the solitary wave solution (Rajaraman, 1982):

$$\phi_o = \tanh\left[\frac{1}{\sqrt{2}}(x - x_o)\right]. \tag{16}$$

The perturbed, linearized equation reads

$$-\frac{d^2\psi}{dx^2} - (6\,\mathrm{sech}^2 x)\psi = (2\omega^2 - 4)\psi. \tag{17}$$

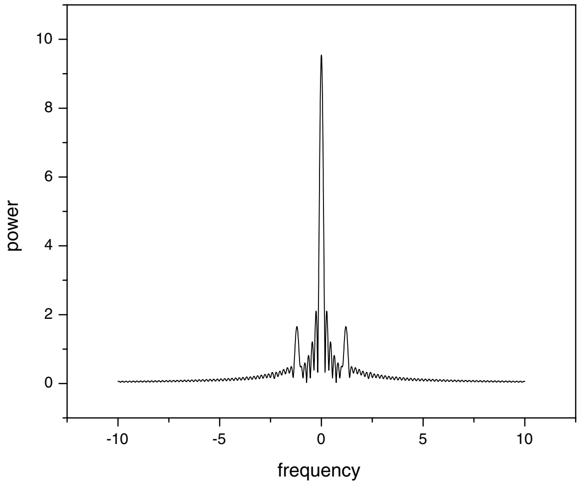

Fig. 2. Fourier transform of the time-dependent solution of the ϕ^4 equation, initially perturbed with $\psi(x)$ and $a = 1$. Note the existence of a non-trivial mode, as compared with Fig. 1.

This equation has two bound eigen-solutions (Lamb, 1980). The symmetrical (ground state) solution corresponds to $\omega = 0$ and the antisymmetric solution has $\omega = \pm\sqrt{3/2}$. The corresponding eigenfunctions are $\psi_1 = a\,\mathrm{sech}\,x$ and $\psi_2 = a\mathrm{sech}x\tanh x$, respectively. In order to further study these modes, we calculated the time evolution of the perturbed solutions, by integrating the full, nonlinear equation of motion, and then Fourier transforming the results. It is seen in Fig. 2 that although initially only the zero (symmetric) mode is excited, the anti-symmetric mode, too, becomes excited, as a result of the nonlinearity of the system.

2.3. Double Sine-Gordon Equation

Consider a modified sine-Gordon equation with the following potential (Riazi and Gharaati, 1998):

$$V(\phi) = (1 - \cos\phi) + \epsilon\,[1 - \cos(2\phi)]\,. \tag{18}$$

The corresponding dynamical equation of motion is

$$\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial t^2} = \sin\phi + 2\epsilon\,\sin(2\phi). \tag{19}$$

The exact static solution of this equation is

$$\phi_o = 4\tan^{-1}\sqrt{\frac{1-A}{1+A}}, \tag{20}$$

where

$$A = \frac{\sinh(\sqrt{4\epsilon+1}x)}{\sqrt{4\epsilon+\cosh^2\sqrt{4\epsilon+1}x}}.$$

By perturbing this solution ($\phi = \phi_o + \psi e^{i\omega t}$) and linearizing the resulting equation, we obtain the Schrödinger-like equation

$$-\frac{d^2\psi}{dx^2} + V(x)\psi = \omega^2\psi, \tag{21}$$

where $V(x)$ is given by

$$V(x) = 32\epsilon\alpha^2 - (32\epsilon - 2)\alpha + 4\epsilon - 1, \tag{22}$$

with

$$\alpha = \frac{\sinh^2\beta x}{4\epsilon + \cosh^2\beta x} \quad \text{and} \quad \beta = \sqrt{4\epsilon+1}.$$

This potential is too complicated to be solved exactly. We first tried asymptotic solutions, and then solved the equation numerically, using a combined variational-finite-difference method, leading to well-behaved solutions. We have found two eigen-solutions. The symmetric and anti-symmetric solutions corresponding to $\epsilon = 0.5$, for $\omega_1 = 0$ and $\omega_2 = 1.243$ are shown in Fig. 3. Solutions of (21) have very sensitive dependencies on ω's, such that if we change ω very slightly, they will diverge very rapidly. The values of the eigenvalues can therefore be pin-pointed easily.

2.4. Double Sinh-Gordon Equation

Let us consider the dynamical equation

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = 4z\sinh(2\phi)(z\cosh(2\phi) - n), \tag{23}$$

where z is a constant, n is an integer and $n > z$ (Khare *et al.*, 1997). The static solution of this equation is

$$\phi_o = \tanh^{-1}\left[\tanh(\Phi_o)\tanh\left(\frac{x-x_o}{\xi}\right)\right] \tag{24}$$

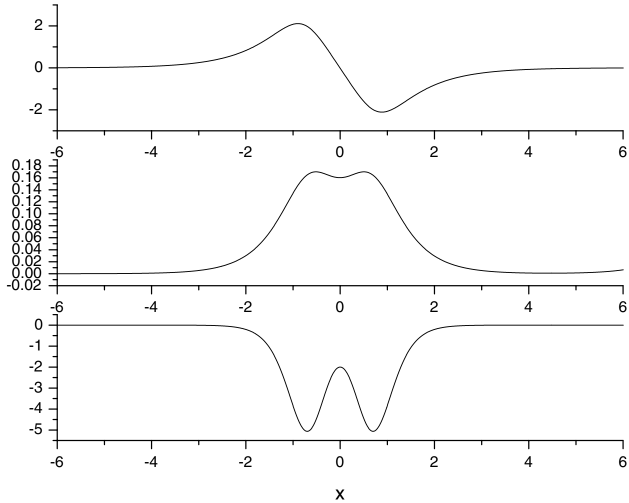

Fig. 3. The linearized potential of the double sine-Gordon equation and the two bound states for $\epsilon = 0.5$. The corresponding frequencies are $\omega = 0$ and 1.243.

where $\xi = [2(n^2 - z^2)]^{-1/2}$ and $\cosh(2\Phi_o) = \frac{n}{z}$. The Schrödinger-like equation which results from small perturbations around this static solution reads:

$$-\psi'' + V(x)\psi = \omega^2\psi, \tag{25}$$

where *V(x)* is given by

$$V(x) = 8z^2\left(\frac{1+3A^2}{1-A^2}\right)^2 - 8zn\left(\frac{1+A^2}{1-A^2}\right). \tag{26}$$

Here, $A = \tanh(\Phi_o)\tanh(\frac{x-x_o}{\xi})$. The ground and four excited states are shown in Fig. 4 for $z = 0.25$ and $n = 1$. The corresponding values of ω are approximately equal to 0.707, 2.29, 3.23, 3.81, and 4.06.

2.5. Modified Sine-Gordon Equation

Consider the following modification of the sine-Gordon equation (Riazi and Gharaati, 1998):

$$\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial t^2} = -\frac{\partial V(\phi)}{\partial\phi}, \tag{27}$$

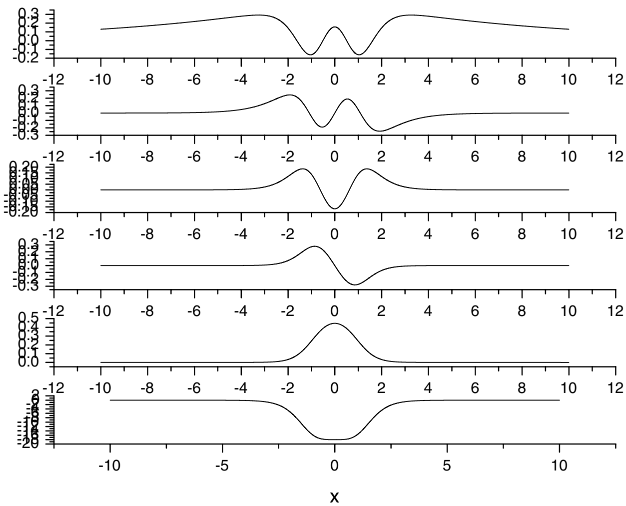

Fig. 4. The linearized potential of the double sinh-Gordon equation and the five bound states for $n = 1$, and $z = 0.25$. The corresponding frequencies are 0.707, 2.29, 3.23, 3.81, and 4.06.

in which

$$V(\phi) = \frac{1 - \cos\phi^s}{s^2\phi^{2s-2}} = \frac{2\sin^2\phi^s/2}{s^2\phi^{2s-2}}. \tag{28}$$

The static soliton-like solution of this equation is given by (Riazi and Gharaati, 1998):

$$\phi_o = [4\tan^{-1} e^x]^{1/s}. \tag{29}$$

The Schrödinger-like equation which results by substituting $\phi = \phi_o + \psi e^{i\omega t}$ in (27), has the following attractive potential

$$V(x) = \frac{2}{s^2} q^{\frac{1-2s}{s}} \left[s\left(\frac{s}{2} - 1\right)\sin q - \frac{1}{2}s^2 q\cos q \right] + \frac{2(1-2s)}{s^2} q^{\frac{2-4s}{s}} \left[(s-1)(1-\cos q) - \frac{1}{2} sq\sin q \right], \tag{30}$$

where $q = 4\tan^{-1} e^x$. For $s = \frac{1}{2}$ it reads:

$$V(x) = 4\,\text{sech}(x) - 2\,\text{sech}(x)\tanh(x) + 4\tan^{-1}(e^x)\,[1 - 2\,\text{sech}(x)]. \tag{31}$$

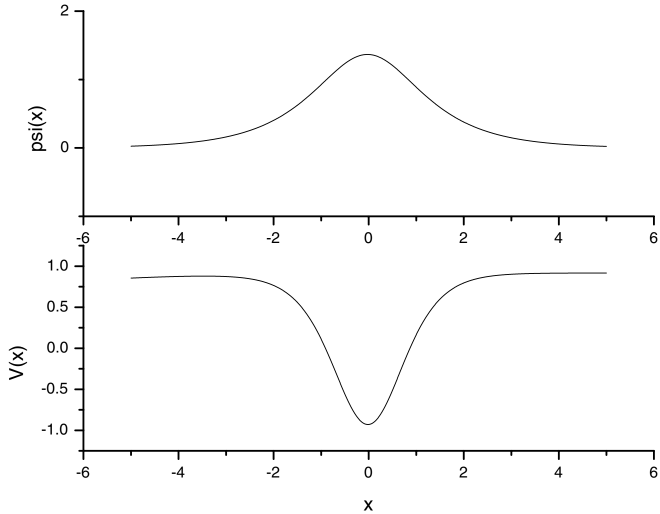

Fig. 5. The potential of the linearized, modified sine-Gordon equation and the first bound state for $s = 1.05$ and $\omega = 0.1268$.

The eigenvalues and eigenfunctions are again computed numerically. The first bound state is shown in Fig. 5. The corresponding value of ω for $s = 1.05$ is 0.1268. Numerical results show that $\omega \to 0$ as $s \to 1$.

3. CONCLUSIONS

In $1 + 1$ dimensions, Schrödinger equation with an attractive potential has at least one bound state (Lamb, 1980). We showed that for the conventional sine-Gordon equation which is an integrable system, one and only one bound state with zero frequency exists, while for non-integrable systems which have topological, solitary solutions, there exists at least one extra bound state with non-vanishing frequency. We therefore, put forward the conjecture that the existence of such bound states are a signature of the non-integrability of the system, at least for the systems considered in the present work. This conjecture is stimulated by the observation that the solitary solutions of non-integrable equations, can absorb energy from the collective translational energy in the process of collisions with other solitary waves or perturbing inhomogeneities. If no other bound states except for the $\omega = 0$ one exist, this channel for energy absorption does not exist and the soliton must leave the interaction area without any deformation.

The study of the internal modes of non-relativistic equations which do not possess static solutions is a more subtle problem. In the case of weakly nonlinear optical media, we know that there are envelope solitons which satisfy the cubic nonlinear Schrödinger (NLS) equation. It is already known that such a system does not possess internal modes (Pelinovsky *et al*., 1998). Pelinovsky *et al*. have studied the internal modes of systems more general than the cubic NLS equation. In particular, they have studied non-integrable equations which have envelope, soliton-like solutions. It is known that such envelope soltary waves show long-lived oscillations which are practically undamped. Pelinovsky *et al*., 1998 showed that the generalized NLS equation possesses localized internal modes similar to those of the topological ϕ^4 system we considered above. The internal mode shows up as a "beating" in the amplitude of the envelope solitary wave. This observation suggests that our conjecture regarding the relationship between the integrability and the existence of non-trivial internal modes might be extended to non-relativistic, envelope solitons.

This conclusion leaves an interesting problem for those working on soliton theory to show-in a rigorous way-whether our conjecture holds generically.

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